Problem Set 10Q2 + 2019ex Q1

Macroeconomics III

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Problem 2

Monetary policy model where the government perfectly controls inflation,

$$\pi_t = m_t$$

Employment is given by:

$$x_t = \theta_t + \pi_t - \pi_t^e, \qquad \pi_t^e = \mathbb{E}_t[\pi_t | \theta_t]$$

Furthermore, the inflation expectations follow,

$$\pi_t^e = \begin{cases} \pi_t^C & \text{if } \pi_{t-1} = \pi_{t-1}^C \\ \pi_t^D & \text{if } \pi_{t-1} \neq \pi_{t-1}^C \end{cases}$$

The social loss function is

$$L(\pi_t, x_t) = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right), \qquad \lambda > 0$$

The intertemporal loss function is

$$\sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t), \qquad \beta \in (0, 1)$$

Quadratic objective $\implies$ Optimal commitment policy is linear (Lecture 10, slide 10)

$$\pi_t = \psi + \psi_\theta \theta_t$$

Problem 2a - Policy - Commitment (1/2)

Compute the equilibrium policy under **commitment** and under discretion

$$\pi_t = \psi + \psi_\theta \theta_t \tag{1}$$

$$x_t = \theta_t + \pi_t - \pi_t^e, \qquad \pi_t^e = \mathbb{E}_t[\pi_t | \theta_t] \tag{2}$$

$$\pi_t^e = \begin{cases} \pi_t^C & \text{if } \pi_{t-1} = \pi_{t-1}^C \\ \pi_t^D & \text{if } \pi_{t-1} \neq \pi_{t-1}^C \end{cases} \tag{3}$$

$$L(\pi_t, x_t) = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right), \qquad \lambda > 0 \tag{4}$$

$$\sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t), \qquad \beta \in (0,1) \tag{5}$$

The government follow a commitment policy that is ex ante optimal. It chooses ψ and ψ_θ to minimise expected loss,

- Find x_t using (1), (2) and (3)
- Insert into $\mathbb{E}_t[L(\pi_t, x_t)]$ and minimize wrt. ψ and ψ_θ

Problem 2a - Policy - Commitment (2/2)

We find x_t by inserting the policy rule and expectations,

$$\begin{aligned} x_t &= \theta_t + \pi_t - \pi_t^e \\ &= \theta_t + \psi + \psi_\theta \theta_t - \mathbb{E}_t[\psi + \psi_\theta \theta_t | \theta_t] \\ &= \theta_t + \psi + \psi_\theta \theta_t - (\psi + \psi_\theta \theta_t) \\ &= \theta_t \end{aligned}$$

We insert this into the expected loss function,

$$\begin{aligned} \mathbb{E}_t \left[L(\pi_t, x_t) \right] &= \mathbb{E}_t \left[\frac{1}{2} \left(\pi_t^2 + \lambda x_t^2 \right) \right] \\ &= \mathbb{E}_t \left[\frac{1}{2} \left((\psi + \psi_\theta \theta_t)^2 + \lambda \theta_t^2 \right) \right] \end{aligned}$$

We can see that $\psi = \psi_\theta = 0$ minimizes the loss function. The optimal policy rule for commitment and the employment is then:

$$\begin{aligned} \pi_t^C &= 0 \\ x_t &= \theta_t \end{aligned}$$

Alternative with FOCs

Problem 2a - Policy - Discretion (1/3)

Compute the equilibrium policy under commitment and under **discretion**

$$x_t = \theta_t + \pi_t - \pi_t^e, \qquad \pi_t^e = \mathbb{E}_t[\pi_t|\theta_t] \tag{6}$$

$$\pi_t^e = \begin{cases} \pi_t^C & \text{if } \pi_{t-1} = \pi_{t-1}^C \\ \pi_t^D & \text{if } \pi_{t-1} \neq \pi_{t-1}^C \end{cases} \tag{7}$$

$$L(\pi_t, x_t) = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right), \qquad \lambda > 0 \tag{8}$$

$$\sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t), \qquad \beta \in (0,1) \tag{9}$$

The government chooses π_t after expectations are formed - the policy must be ex post optimal. Forget about the optimal policy rule.

- Insert x_t into $L(\pi_t, x_t)$
- Minimize the loss-function wrt. π_t and isolate π_t and denote it π_t^D
- Take expectations of π_t^D, which is then π_t^e
- Insert π_t^e into π_t^D to get the optimal policy

Problem 2a - Policy - Discretion (2/3)

The loss-function is given by,

$$\begin{aligned} L(\pi_t, x_t) &= \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right) \\ &= \frac{1}{2}\left(\pi_t^2 + \lambda(\theta_t + \pi_t - \pi_t^e)^2\right) \end{aligned}$$

The first-order condition

$$\frac{dL\left(\pi_t, x_t\right)}{d\pi_t} = 0 \qquad \Longrightarrow \qquad \begin{aligned} \pi_t^D &= -\lambda\left(\theta_t + \pi_t^D - \pi_t^e\right) \\ \pi_t^D(1+\lambda) &= \lambda(\pi_t^e - \theta_t) \\ \pi_t^D &= \frac{\lambda}{1+\lambda}(\pi_t^e - \theta_t) \end{aligned}$$

Hence, we have found the optimal policy for given expectations.

Problem 2a - Policy - Discretion (3/3)

We find $\mathbb{E}\left[\pi_t^D | \theta_t\right] = \pi_t^e$ by taking expectations of π_t^D and isolate π_t^e,

$$\begin{aligned}
\pi_t^e &= \frac{\lambda}{1+\lambda}\mathbb{E}_t\left[\pi_t^e - \theta_t | \theta_t\right] \\
\frac{1+\lambda}{1+\lambda}\pi_t^e &= \frac{\lambda}{1+\lambda}\left(\pi_t^e - \theta_t\right) \\
\frac{1}{1+\lambda}\pi_t^e &= -\frac{\lambda}{1+\lambda}\theta_t \\
\pi_t^e &= -\lambda\theta_t
\end{aligned}$$

We insert this into the optimal policy function,

$$\begin{aligned}
\pi_t^D &= \frac{\lambda}{1+\lambda}(\pi_t^e - \theta_t) \\
\pi_t^D &= -\frac{\lambda}{1+\lambda}(\lambda\theta_t + \theta_t) \\
\pi_t^D &= -\lambda\theta_t \\
x_t^D &= \theta_t + \pi_t - \pi_t^e = \theta_t - \lambda\theta_t + \lambda\theta_t = \theta_t
\end{aligned}$$

Problem 2b - Optimal Deviation

Compute the optimal deviation for a government that is expected to play π^C.

- What are the benefits and the cost of such a deviations?
- For what value of θ is it optimal for the government to stick to the commitment rule?

The government initially runs a credible monetary policy rule under commitment, why the expectations are formed accordingly. By deviating from the announced rule, the government can surprise the agents and benefit from it short-term.

By doing so, there will be a cost since it will lose credibility.

Problem 2b - Optimal Deviation

Calculate what the benefit is from deviating from commitment policy. Remember that $\pi^C = 0$ and thus $\pi^e = 0$ if commitment is followed.

That is calculate

$$B = L\left(\pi_t^C, x_t^C\right) - L\left(\pi_t^*, x_t^*\right)$$

Where π^* and x_t^* is the inflation and employment if it deviates.

Remember: Under commitment

$$\pi_t^C = 0 \qquad\qquad x_t^C = \theta_t$$

Optimal inflation under discretion for given expectations

$$\pi_t^D = \frac{\lambda}{1+\lambda}(\pi_t^e - \theta_t)$$

When expectations are in place $\pi^e = \pi^D$ then

$$\pi_t^C = -\lambda\theta_t \qquad\qquad x_t^C = \theta_t$$

Problem 2b - Optimal Deviation - Benefit (1/2)

Under commitment, we know that $\pi^C = 0$, why $\pi^e = 0$. The optimal policy is given by the FOC in question a.

$$\pi_t^D = \frac{\lambda}{1+\lambda}(\pi_t^e - \theta_t) = -\frac{\lambda}{1+\lambda}\theta_t$$

Which gives the following employment,

$$x_t = \theta_t - \frac{\lambda}{1+\lambda}\theta_t = \frac{1}{1+\lambda}\theta_t$$

The benefit is given by the difference in the loss-functions with and without the deviation,

$$\begin{aligned} B &= L\left(\pi_t^C, x_t^C\right) - L\left(\pi_t^*, x_t^*\right) \\ &= \frac{1}{2}\left(\lambda\theta_t^2 - \left(\frac{\lambda}{1+\lambda}\theta_t\right)^2 - \lambda\left(\frac{1}{1+\lambda}\theta_t\right)^2\right) \end{aligned}$$

Problem 2b - Optimal Deviation - Benefit (2/2)

We rewrite the benefit of the deviation.

$$
\begin{aligned}
B &= \frac{1}{2}\left(\lambda\theta_t^2 - \left(\frac{\lambda}{1+\lambda}\theta_t\right)^2 - \lambda\left(\frac{1}{1+\lambda}\theta_t\right)^2\right) \\
&= \frac{1}{2}\left(\frac{(1+\lambda)^2\lambda}{(1+\lambda)^2} - \frac{\lambda^2}{(1+\lambda)^2} - \frac{\lambda}{(1+\lambda)^2}\right)\theta_t^2 \\
&= \frac{1}{2}\left(\frac{(1+\lambda)^2\lambda - \lambda(1+\lambda)}{(1+\lambda)^2}\right)\theta_t^2 \\
&= \frac{1}{2}\left(\frac{(1+\lambda)\lambda - \lambda}{(1+\lambda)}\right)\theta_t^2 \\
&= \frac{1}{2}\frac{\lambda^2}{(1+\lambda)}\theta_t^2
\end{aligned}
$$

Hence, we have found the benefit from deviating from the commitment policy.

Problem 2b - Optimal Deviation - Cost (1/3)

The government can implement the deviation through two different policies,

1. Deviate for only one period. Go back to the commitment policy right after the deviation and incur a loss in that period. $\pi^C_{t-1}, \pi^D_t, \pi^C_{t+1} \ldots$
2. Play a trigger-strategy such that is deviates and then follow a discretion rule in the following periods. $\pi^C_{t-1}, \pi^D_t, \pi^D_{t+1} \ldots$

Calculate the costs of deviating. The expected costs will be:

$$C_1 = \beta \mathbb{E}_t \Big[L\left(\pi^C_{t+1}, x^*_{t+1}\right) - L\left(\pi^C_{t+1}, x^C_{t+1}\right) \Big]$$

$$C_2 = \sum_{i=1}^{\infty} \beta^i \mathbb{E}_t \left[L\left(\pi^D_{t+i}, x^D_{t+i}\right) - L\left(\pi^C_{t+i}, x^C_{t+i}\right) \right]$$

By following the first strategy, the employment in the following period will be,

$$x^*_{t+1} = \theta_{t+1} + \pi^C_{t+1} - \pi^e_{t+1} = \theta_{t+1} + 0 + \lambda\theta_{t+1} = (1+\lambda)\theta_{t+1}$$

Problem 2b - Optimal Deviation - Cost (2/3)

We rewrite the cost

$$
\begin{aligned}
C_1 &= \beta \mathbb{E}_t \Big[L\left(\pi^C_{t+1}, x^L_{t+1}\right) - L\left(\pi^C_{t+1}, x^C_{t+1}\right) \Big] \\
&= \frac{1}{2}\beta \mathbb{E}_t \Big[0^2 + \lambda(1+\lambda)^2 \theta^2_{t+1} - 0^2 - \lambda \theta^2_{t+1} \Big] \\
&= \frac{1}{2}\beta \lambda \mathbb{E}_t \Big[(1+\lambda)^2 \theta^2_{t+1} - \theta^2_{t+1} \Big] \\
&= \frac{1}{2}\beta \lambda \mathbb{E}_t \Big[(1+\lambda^2+2\lambda) \theta^2_{t+1} - \theta^2_{t+1} \Big] \\
&= \frac{1}{2}\beta \lambda \mathbb{E}_t \Big[\lambda(2+\lambda) \theta^2_{t+1} \Big] \\
&= \frac{1}{2}\beta (2+\lambda)\lambda^2 \mathbb{E}_t \left[\theta^2_{t+1} \right]
\end{aligned}
$$

Hence, we have found the cost of strategy 1.

Problem 2b - Optimal Deviation - Cost (3/3)

We turn to strategy 2 where the government turns to discretionary monetary policy,

$$
\begin{aligned}
C_2 &= \sum_{i=1}^{\infty} \beta^i \mathbb{E}_t \left[L\left(\pi_{t+i}^D, x_{t+i}^D\right) - L\left(\pi_{t+i}^C, x_{t+i}^C\right)\right] \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \beta^i \mathbb{E}_t \Big[\underbrace{\lambda^2 \theta_{t+i}^2}_{(\pi_t^D)^2 = (-\lambda\theta_t)^2} + \lambda \theta_{t+i}^2 - 0^2 - \lambda \theta_{t+i}^2 \Big] \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \beta^i \mathbb{E}_t \left[\lambda^2 \theta_{t+i}^2\right]
\end{aligned}
$$

We use that $\mathbb{E}_t[\theta_{t+1}] = \mathbb{E}_t[\theta_{t+i}]$ for $i \geq 1$ since it is i.i.d.

$$
\begin{aligned}
&= \frac{1}{2} \mathbb{E}_t \left[\lambda^2 \theta_{t+1}^2\right] \sum_{i=1}^{\infty} \beta^i \\
&= \frac{\beta}{1-\beta} \frac{1}{2} \lambda^2 \mathbb{E}_t \left[\theta_{t+1}^2\right]
\end{aligned}
$$

Hence, we have the cost of strategy 2.

Problem 2b - Optimal Deviation (1/3)

We now have the benefits from deviating and the expected costs of each strategy Since the government knows the expected costs, it can choose the strategy with the lowest cost. Hence, the government will deviate when $B > \min(C_1, C_2)$. We have two scenarios

For $C_1 < C_2$

$$B > C_1 \qquad \Leftrightarrow \qquad \begin{aligned} \frac{1}{2}\frac{\lambda^2}{(1+\lambda)}\theta_t^2 &> \frac{1}{2}\beta(2+\lambda)\lambda^2\mathbb{E}_t\left[\theta_{t+1}^2\right] \\ \theta_t^2 &> (1+\lambda)(2+\lambda)\beta E\left[\theta_{t+1}^2\right] \end{aligned}$$

For $C_1 > C_2$

$$B > C_2 \qquad \Leftrightarrow \qquad \begin{aligned} \frac{1}{2}\frac{\lambda^2}{(1+\lambda)}\theta_t^2 &> \frac{\beta}{1-\beta}\frac{1}{2}\lambda^2\mathbb{E}_t\left[\theta_{t+1}^2\right] \\ \theta_t^2 &> (1+\lambda)\frac{\beta}{1-\beta}E\left[\theta_{t+1}^2\right] \end{aligned}$$

Problem 2b - Optimal Deviation (2/3)

For $C_1 < C_2$

$$B > C_1 \quad \Leftrightarrow \quad \theta_t^2 > (1+\lambda)(2+\lambda)\beta E\left[\theta_{t+1}^2\right]$$

For $C_1 > C_2$

$$B > C_2 \quad \Leftrightarrow \quad \theta_t^2 > (1+\lambda)\frac{\beta}{1-\beta}E\left[\theta_{t+1}^2\right]$$

θ_t is the potential level of employment (output) from the potential.

The government will benefit from deviating when the current potential employment level is far from its mean.

The required level of θ^2 is increasing in β since β only affects the costs and not the benefits.

Problem 2b - Optimal Deviation (3/3)

Finally, we investigate when $C_1 > C_2$,

$$
\begin{aligned}
C_1 &> C_2 \\
\frac{1}{2}\beta(2+\lambda)\lambda^2\mathbb{E}_t\left[\theta_{t+1}^2\right] &> \frac{\beta}{1-\beta}\frac{1}{2}\lambda^2\mathbb{E}_t\left[\theta_{t+1}^2\right] \\
2+\lambda &> \frac{1}{1-\beta} \\
1-\beta &> \frac{1}{2+\lambda} \\
1-\frac{1}{2+\lambda} &> \beta \\
\frac{1+\lambda}{2+\lambda} &> \beta
\end{aligned}
$$

When firms are impatient (low β) the relative cost of C_1 where the costs are incurred in the first period, are higher than C_2 where the costs are distributed over the entire horizon.

January 2019 Q1

OLG economy with depreciation $\delta = 1$. The following holds,

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} \tag{10}$$

$$1 + r_t = \alpha A k_t^{\alpha - 1} \tag{11}$$

$$w_t = (1-\alpha) A k_t^{\alpha} \tag{12}$$

The individuals in the economy face the following maximization problem,

$$\max_{c_{1t},\, c_{2t+1}} U_t = \ln(c_{1t}) + \frac{1}{1+\rho} \ln(c_{2t+1})$$

$$\text{s.t.} \qquad c_{1t} + s_t = (1-\tau) w_t$$

$$c_{2t+1} = s_t(1 + r_{t+1}) + d_{t+1}$$

January 2019 Q1a (1/4)

1. Set up and solve the individualís problem of optimal intertemporal allocation of resources.
2. Derive the Euler equation.
3. Show that individual saving behavior is characterized by

$$s_t = \frac{1}{2+\rho} w_t (1-\tau) - \tau \frac{1+\rho}{2+\rho} \frac{1}{1+r_{t+1}} w_{t+1}$$

Euler equation:
Combine the budget constraints to get one budget constraint of the form,

$$c_{1t} + \frac{c_{2t+1}}{(1+r_{t+1})} = ...$$

Find FOCs wrt. c_{1t} and c_{2t+1}. Combine to get the Euler equation.

Savings: Plug the Euler equation into the budget constraints to find s_t

January 2019 Q1a (2/4)

Combine the budget constraints,

$$
\begin{aligned}
c_{1t} &= w_t(1-\tau) - s_t \\
\frac{c_{2t+1}}{1+r_{t+1}} &= s_t + \frac{1}{1+r_{t+1}}\tau w_{t+1} \\
c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} &= w_t(1-\tau) + \frac{1}{1+r_{t+1}}\tau w_{t+1}
\end{aligned}
$$

The maximisation problem is then given by,

$$
\begin{aligned}
L = &\ln(c_{1t}) + \frac{1}{1+\rho}\ln(c_{2t+1}) \\
&+ \lambda\left(w_t(1-\tau) + \frac{1}{1+r_{t+1}}\tau w_{t+1} - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}}\right)
\end{aligned}
$$

FOC wrt c_{1t} and c_{2t+1}

$$
\frac{1}{c_{1t}} = \lambda \qquad\qquad \frac{1}{1+\rho}\frac{1}{c_{2t+1}} = \frac{\lambda}{1+r_{t+1}}
$$

January 2019 Q1a (3/4)

Combining the FOCs yields the Euler equation:

$$\frac{c_{1t}(1 + r_{t+1})}{1 + \rho} = c_{2t+1}$$

The intuition is more or less the same as the intuition from the standard Euler equation in the Ramsey model.

We see that the Euler equation depends on the patience $\frac{1}{1+\rho}$ such that an increase in ρ leads to higher c_{1t} relative to c_{2t+1}. This is because agents become less patient when ρ increases (note the difference between ρ and β).

On the other hand, an increase in r_{t+1} will put an upwards pressure on c_{2t+1}

January 2019 Q1a (4/4)

We then plug the Euler equation into the budget constraints to retrieve the savings function:

$$
\begin{aligned}
s_t(1+r_{t+1}) &= c_{2t+1} - \tau w_{t+1} \\
s_t(1+r_{t+1}) &= \frac{1+r_{t+1}}{1+\rho} c_{1t} - \tau w_{t+1} \\
s_t &= \frac{1}{1+\rho} c_{1t} - \frac{1}{1+r_{t+1}} \tau w_{t+1} \\
s_t &= \frac{1}{1+\rho} \big(w_t(1-\tau) - s_t \big) - \frac{1}{1+r_{t+1}} \tau w_{t+1} \\
s_t \left(1 + \frac{1}{1+\rho}\right) = s_t \frac{2+\rho}{1+\rho} &= \frac{1}{1+\rho} w_t(1-\tau) - \frac{1}{1+r_{t+1}} \tau w_{t+1} \\
s_t &= \frac{1}{2+\rho} w_t(1-\tau) - \frac{1+\rho}{2+\rho} \frac{1}{1+r_{t+1}} \tau w_{t+1}
\end{aligned}
$$

Hence, savings depends positively on the disposable income when young, $(1-\tau)w_t$ and negatively on the contributions received when old, τw_{t+1}.

January 2019 Q1b (1/3)

Show that the capital accumulation equation that gives k_{t+1}, as a function of k_t, is given by

$$k_{t+1} = \frac{1}{1 + \frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left[\frac{(1-\alpha)(1-\tau)}{2+\rho}Ak_t^{\alpha}\right]$$

Show also that the steady state capital level is given by

$$\bar{k} = \left[\frac{1}{1 + \frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left(\frac{1}{2+\rho}(1-\alpha)A(1-\tau)\right)\right]^{\frac{1}{1-\alpha}}$$

Use that

$$\begin{aligned} k_{t+1} &= s_t \\ w_t &= (1-\alpha)Ak_t^{\alpha} \\ r_t &= \alpha A k_t^{\alpha-1} \end{aligned}$$

January 2019 Q1b (2/3)

We start by using $s_t = k_{t+1}$

$$k_{t+1} = \frac{1}{2+\rho} w_t(1-\tau) - \frac{1+\rho}{2+\rho}\frac{1}{1+r_{t+1}}\tau w_{t+1}$$

We insert the wage and interest rate expressions

$$k_{t+1} = \frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau) - \frac{1+\rho}{2+\rho}\frac{(1-\alpha)Ak_{t+1}^{\alpha}}{\alpha Ak_{t+1}^{\alpha-1}}\tau$$

$$k_{t+1} = \frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau) - \frac{1+\rho}{2+\rho}\frac{(1-\alpha)k_{t+1}}{\alpha}\tau$$

We isolate k_{t+1} on the left hand side,

$$k_{t+1}\left(1 + \frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau\right) = \frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau)$$

Finally, we isolate k_{t+1} fully,

$$k_{t+1} = \frac{1}{1+\frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left[\frac{(1-\alpha)(1-\tau)}{2+\rho}Ak_t^{\alpha}\right]$$

January 2019 Q1b (3/3)

We have that the capital accumulation follows

$$k_{t+1} = \frac{1}{1+\frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left[\frac{(1-\alpha)(1-\tau)}{2+\rho}Ak_t^{\alpha}\right]$$

In steady-state we have $k_t = k_{t+1} = \bar{k}$,

$$\begin{aligned}
\bar{k} &= \frac{1}{1+\frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left[\frac{(1-\alpha)(1-\tau)}{2+\rho}A\bar{k}^{\alpha}\right] \\
\bar{k}^{1-\alpha} &= \left[\frac{1}{1+\frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left(\frac{1}{2+\rho}(1-\alpha)A(1-\tau)\right)\right] \\
\bar{k} &= \left[\frac{1}{1+\frac{1+\rho}{2+\rho}\frac{(1-\alpha)}{\alpha}\tau}\left(\frac{1}{2+\rho}(1-\alpha)A(1-\tau)\right)\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

January 2019 Q1c (1/3)

The problem is now

$$\max_{c_{1t},\ c_{2t+1}} U_t = \ln(c_{1t}) + \frac{1}{1+\rho}\ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} + s_t = (1-\tau)w_t$$

$$c_{2t+1} = (s_t + \tau w_t)(1 + r_{t+1})$$

Show that the new steady-state capital-per-worker, which is denoted by $\bar{k}'$, is such that

$$\bar{k}' = \left[\left(\frac{1}{2+\rho}\right)(1-\alpha)A\right]^{\frac{1}{1-\alpha}}$$

for $t \geq T$ and where $\tau < \frac{1}{2+\rho}$ is implicitly imposed.

January 2019 Q1c (2/3)

We proceed by substituting in the budget constraints and maximize wrt. s_t as per usual,

$$U = \ln((1-\tau)w_t - s_t) + \frac{1}{1+\rho}\ln\left((s_t + \tau w_t)(1+r_{t+1})\right)$$

FOC wrt s_t:

$$\begin{aligned}
\frac{1}{(1-\tau)w_t - s_t} &= \frac{1}{1+\rho}\frac{1}{s_t+\tau w_t} \\
s_t + \tau w_t &= \frac{1}{1+\rho}\left((1-\tau)w_t - s_t\right) \\
s_t\left(\frac{2+\rho}{1+\rho}\right) &= \frac{1}{1+\rho}w_t - \frac{2+\rho}{1+\rho}\tau w_t \\
s_t &= \left(\frac{1}{2+\rho} - \tau\right)w_t
\end{aligned}$$

January 2019 Q1c (3/3)

We have the following savings.

$$s_t = \left(\frac{1}{2+\rho} - \tau \right) w_t$$

We use $w_t = (1-\alpha)Ak_t^{\alpha}$ and $s_t = k_{t+1}$

$$\begin{aligned} k_{t+1} &= \left(\frac{1}{2+\rho} - \tau \right) w_t \\ &= \left(\frac{1}{2+\rho} - \tau \right) (1-\alpha)Ak_t^{\alpha} \end{aligned}$$

In the steady state, capital is constant at $\bar{k}' = k_t = k_{t+1}$

$$\bar{k}' = \left[\left(\frac{1}{2+\rho} - \tau \right) (1-\alpha)A \right]^{\frac{1}{1-\alpha}}$$

January 2019 Q1d

Is the older generation at time T better off or worse off after the social security system has been changed?

The old are worse off since they no longer get any pension as the change happens before the savings decision is formulated.

Problem 2a - Policy - Commitment - Alternative (1/2)

We find x_t by inserting the policy rule and expectations,

$$\begin{aligned} x_t &= \theta_t + \pi_t - \pi_t^e \\ &= \theta_t + \psi + \psi_\theta \theta_t - \mathbb{E}_t[\psi + \psi_\theta \theta_t | \theta_t] \\ &= \theta_t + \psi + \psi_\theta \theta_t - (\psi + \psi_\theta \theta_t) \\ &= \theta_t \end{aligned}$$

We insert this into the expected loss function,

$$\begin{aligned} \mathbb{E}_t\left[L(\pi_t, x_t)\right] &= \mathbb{E}_t\left[\frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right)\right] \\ &= \mathbb{E}_t\left[\frac{1}{2}\left((\psi + \psi_\theta \theta_t)^2 + \lambda \theta_t^2\right)\right] \\ &= \mathbb{E}_t\left[\frac{1}{2}\left(\psi^2 + \psi_\theta^2 \theta_t^2 + 2\psi\psi_\theta \theta_t + \lambda \theta_t^2\right)\right] \\ &= \frac{1}{2}\Big(\psi^2 + \psi_\theta^2 \mathbb{E}_t[\theta_t^2] + 2\psi\psi_\theta \mathbb{E}_t[\theta_t] + \lambda \mathbb{E}_t[\theta_t^2]\Big) \end{aligned}$$

Problem 2a - Policy - Commitment - Alternative (2/2)

It is the expectation of θ since the policy rule is announced before the realisation of θ.

$$\mathbb{E}_t\left[L(\pi_t, x_t)\right] = \frac{1}{2}\Big(\psi^2 + \psi_\theta \mathbb{E}_t[\theta_t^2] + 2\psi\psi_\theta \mathbb{E}_t[\theta_t] + \lambda \mathbb{E}_t[\theta_t^2]\Big)$$

FOC wrt. ψ

$$\psi = \psi_\theta \mathbb{E}_t[\theta_t]$$

FOC wrt. ψ_θ

$$\psi_\theta \mathbb{E}_t[\theta_t^2] = \psi \mathbb{E}_t[\theta]$$

From this, it is obvious that $\psi = \psi_\theta = 0$ satisfies both FOCs. Otherwise, combining the FOCs:

$$\psi_\theta \mathbb{E}_t[\theta_t^2] = \psi_\theta \mathbb{E}_t[\theta_t]^2$$
$$\psi_\theta \underbrace{\left(\mathbb{E}_t[\theta_t^2] - \mathbb{E}_t[\theta_t]^2\right)}_{\equiv var(\theta_t) > 0} = 0$$

Hence, $\psi_\theta = 0$ and thus $\psi = 0$. Back